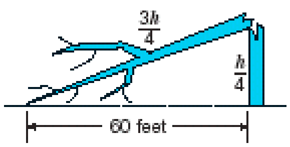
XIX Copa Eugene Francis

Problema 1) Un rayo golpea un árbol a una cuarta parte de su altura, a partir de su tronco y rompió el árbol de modo que su copa cayó 60 pies de su base, como se muestra en la figura. ¿Cuán alto era el árbol originalmente?

Lightning hit a tree one-fourth of the distance up the trunk from the ground and broke the tree so that its top landed 60 feet from its base, as shown in the figure. How tall was the tree originally?

Solution



Approximately 84.85 feet tall.

Let h be the height, in feet, of the tree before it fell. Then $(h/4)^2 + 60^2 = (3h/4)^2$, so

$h^2/16 + 3600 = 9h^2/16$.

$h^2/2 = 3600$,

so $h = \sqrt{7200}$, and the original height of the tree equals $60\sqrt{2}$, or approximately 84.85 feet.

XIX Copa Eugene Francis

Problema 2) Los habitantes de Pluto usan los mismos operadores matemáticos que nosotros

(+, −, etc.). Ellos además usan un operador, @, que nosotros no conocemos. Las siguientes son ciertas para cualesquiera números reales x y y. ¿Cuál es el valor de 12 @ 5?

Pluto's inhabitants use the same mathematical operators that we do (+, −, etc.). They also use and operator, @, that we do not know. The following are true for any real numbers x and y. What is the value of 12 @ 5?

$$x @ 0 = x$$
$$x @ y = y @ x$$
$$(x + 1) @ y = (x @ y) + y + 1$$

Solution

77. We have

$$\begin{aligned} 12 @ 5 &= 5 @ 12 \\ &= (4 @ 12) + 13 \\ &= (3 @ 12) + 13 + 13 \\ &= (3 @ 12) + 26. \end{aligned}$$

Continuing,

$$\begin{aligned} (3 @ 12) + 26 &= (2 @ 12 + 39) \\ &= (1 @ 12) + 52 \\ &= (0 @ 12) + 65 \\ &= (12 @ 0) + 65 \\ &= 77. \end{aligned}$$

XIX Copa Eugene Francis

Problema 3) El producto de tres enteros impares consecutivos reducido por 23 es 99 menos que el cubo de la suma del número menor y 2. Calcular el promedio de estos tres enteros.

The product of three consecutive odd integers reduced by 23 is 99 less than the cube of the sum of the smallest number and 2. Compute the mean of these three integers.

Solution:

Let the three integers be: $x - 2, x, x+2$

We need to solve:

$(x - 2)x(x+2) - 23 = (x - 2 + 2)^3 - 99$

$x^3 - 4x - 23 = x^3 - 99$

$4x = 76$

$x = 19$

Notice that the mean of these three numbers is: $\frac{x - 2 + x + x + 2}{3} = x$

The mean is: 19

XIX Copa Eugene Francis

Problema 4) Un número de tres dígitos es seleccionado al azar de entre todos los números de tres dígitos posibles formados por los dígitos 1, 2, 3, 4, y 6. ¿Cuál es la probabilidad que el número seleccionado sea un número par menor que 500 sin repetición de sus dígitos?

A three-digit number is drawn at random from all possible 3-digit numbers formed by the digits 1, 2, 3, 4, and 6. What is the probability that the number drawn is an even number less than 500 containing no digit more than once?

Solution

The corresponding probability is 0.24

There are $5 \times 5 \times 5$, or 125, different 3-digit numbers that can be formed from the 5 given digits. The event requires that the last digit be even and the value of the three-digit number be less than 500.

If a 6 is in the units position there are $4 \times 3 \times 1 = 12$ different-digit numerals of this type.

If a 2 or a 4 is in the units position, there are $3 \times 3 \times 2 = 18$ different-digit numerals of this type.

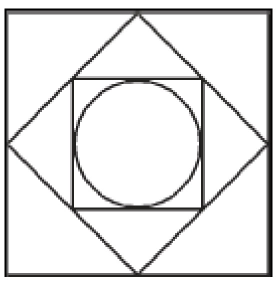
The event therefore can occur in: $12 + 18 = 30$ different ways

The probability of the event is $30/125 = 6/25 = .24$.

XIX Copa Eugene Francis

Problema 5) Los puntos medios de los lados del cuadrado mayor son unidos para formar un nuevo cuadrado. Este proceso se repite usando el nuevo cuadrado, y un círculo se inscribe en este cuadrado menor. Si el área del cuadrado mayor es $\frac{16}{\pi}$, encuentre el área del círculo.

The midpoints of the sides of the largest square are joined to form a new square. The process is repeated using the new square, and a circle is inscribed in this smallest square. If the area of the largest square is $\frac{16}{\pi}$, *find the area of the circle.*



Solution:

Since the largest square has an area of $\frac{16}{\pi}$, the area of the smallest square is $\frac{4}{\pi}$.

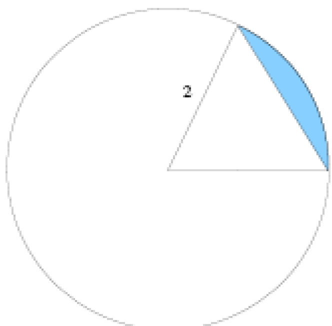
The side of that smallest square is: $\frac{2}{\sqrt{\pi}}$, which equals the diameter of the circle.

Therefore, the radius of the circle is $\frac{1}{\sqrt{\pi}}$ and its area is: $\pi\left(\frac{1}{\sqrt{\pi}}\right)^2 = 1$

XIX Copa Eugene Francis

Problema 6) El siguiente triángulo equilátero tiene uno de sus vértices en el centro del círculo, determine el área de la región sombreada.

The following equilateral triangle has one of its vertices in the center of the circle, determine the area of the shaded region.



Solución:

Como el triángulo es equilátero con lado = 2

su altura es: $\sqrt{3}$

El área del triángulo es: $\frac{1}{2}(2)\sqrt{3} = \sqrt{3}$

El área del sector circular que contiene el triángulo es:

$$A_{sector} = \frac{1}{2}(\frac{\pi}{3})2^2 = \frac{2\pi}{3}$$

(puesto que el sector circular tiene un ángulo de $\frac{\pi}{3}$ (el triángulo es equilátero)

y el radio del círculo es 2

Luego

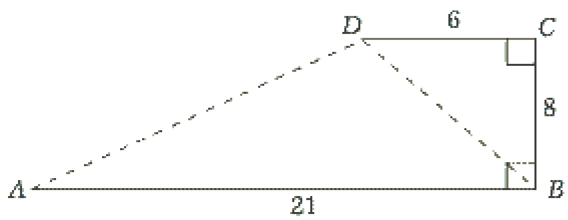
$$A_{sombreada} = \frac{2\pi}{3} - \sqrt{3}$$

XIX Copa Eugene Francis

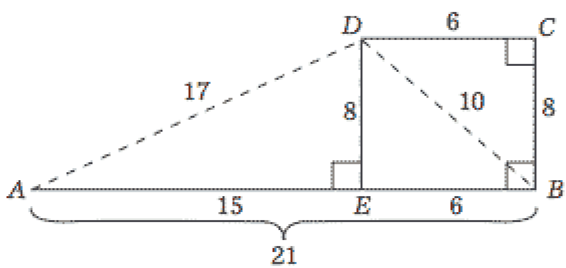
Problema 7

¿Cuál es la suma de las distancias AD y BD en la figura que se muestra?

What is the sum of the distances AD and BD in the figure shown?



Solution



The sum of the distances is 27.

By the Pythagorean theorem, $BD = 10$.

Let E be a point on AB with $\angle DEA = 90°$.

Then $DE = 8$, and $AE = 21 - 6 = 15$.

By the Pythagorean theorem, $AD = 17$.

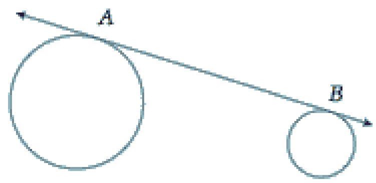
Hence, $10 + 17 = 27$.

XIX Copa Eugene Francis

Problema 8

Dos círculos con radios 8 cm y 13 cm respectivamente, tienen una tangente externa común que determina un segmento AB de largo 12 cm, entre los puntos de tangencia. ¿Cuál es la distancia entre los centros de los círculos?

Two circles with radii of 8 cm and 3 cm, respectively, have a common external tangent that determines a segment AB of length 12 cm between the points of tangency. What is the distance between the centers of the circles?



Solution

The distance between the centers of the circles is 13 cm.

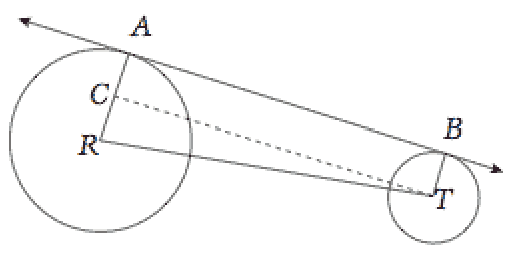
AB is a common external tangent to circles *R* and *T*.

RA and *TB* are perpendicular to *AB* at the points of tangency.

Construct a line perpendicular to *RA* from point *T*, locating point *C*.

ΔRCT is a right triangle having legs *TC* (12 cm) and *RC* (8 – 3 = 5 cm).

Therefore, $(RT)^2 = 5^2 + 12^2 = 169$ and *RT* = 13 cm.



XIX Copa Eugene Francis

Problema 9) Suponga que 10 cartas, de las cuales cinco son rojas y cinco son verdes, son colocadas al azar en 10 sobres, de los cuales cinco son rojos y cinco son verdes. Determine la probabilidad que exactamente dos sobres contengan una carta con su mismo color.

Suppose that 10 cards, of which five are red and five are green, are placed at random in 10 envelopes, of which five are red and five are green. Determine the probability that exactly two envelopes will contain a card with a matching color

Solution:

There are 10! ways of assigning the cards into the envelopes, if no restriction.

For the corresponding event we have:

First, notice that this can only happen if we have 1 red and 1 green coincidence, because if we impose two coincidences of the same color there will be automatically more than 2 coincidences.

Therefore, For the corresponding event we have:

task 1: choosing the red envelope for the red card: C(5,1)=5 ways

task 2: choosing the green envelope for the green card: C(5,1)=5 ways

task3: Assigning the cards within the RED envelopes: 5 (1 out of 5 red cards) 5 4 3 2 (the corresponding green cards) =5*(5!) ways

task 4: Assigning the cards within the GREEN envelopes: 1 (1 remaining green card) 4 3 2 1 (the corresponding red cards) = 4! ways

The corresponding probability is: $\dfrac{5*5**5*(5!)*(4!)}{10!} = \dfrac{(5^4)*4!*4!}{10!}$ =0.099

XIX Copa Eugene Francis

Problema 10

Demuestre que $\left(\frac{3+\sqrt{17}}{2}\right)^n + \left(\frac{3-\sqrt{17}}{2}\right)^n$ es impar para todo número natural n

Proof that $\left(\frac{3+\sqrt{17}}{2}\right)^n + \left(\frac{3-\sqrt{17}}{2}\right)^n$ is odd for all natural number n

Solution:

Let $a = \frac{3+\sqrt{17}}{2}$ and $b = \frac{3-\sqrt{17}}{2}$

By strong induction:

For n = 1:

$a + b = 3$;odd

Assume its true for all integers $\leq n$

Consider: $a^{n+1} + b^{n+1}$

$a^{n+1} + b^{n+1} = a^{n+1} + b^{n+1} + a^n b - a^n b + ab^n - ab^n$

$= a^n (a+b) + b^n (b+a) - ab(a^{n-1} + b^{n-1})$

$= (a^n + b^n)(a+b) - ab(a^{n-1} + b^{n-1})$

$(a^n + b^n)$ and $(a^{n-1} + b^{n-1})$ are odd by hypothesis

$(a+b) = 3$

$ab = -2$ (even)

Hence:

$a^{n+1} + b^{n+1} = odd(odd) - even(odd)$

$= odd$

Desempate 1

Se observa el comportamiento reproductivo de una bacteria, cada 15 minutos cada bacteria puede morir, no reproducirse, dividirse en dos o dividirse en tres, con probabilidad ¼ para cada caso. ¿Cuál es la probabilidad de que la población de bacterias se extinga?

We observe the reproductive behavior of one bacteria. Every 15 minutes, each bacteria can die, do nothing, split into 2 or split into 3 bacteria with probability ¼ for each case. What is the probability that the bacteria population eventually dies out?

Solution:

Let p be the probability that just one bacteria eventually dies out, and all its descendants.
The probability of n bacteria all dying out is p^n.

After the first turn there are four possibilities for the number of bacteria left, 0, 1, 2, 3 with probabilities of eventually dying out of 1, p, p^2, and p^3.

With each outcome being equally likely the probability of all bacteria eventually dying out is $1/4*[1 + p + p^2 + p^3]$.

So $p = 1/4*[1 + p + p^2 + p^3]$

$1 - 3p + p^2 + p^3 = 0$.

This reduces to $(p - 1)(p^2 + 2p - 1) = 0$.

The solutions for p are 1, $-1+\sqrt{2}$ and $-1-\sqrt{2}$

The only one which satisfies the constraints of the problem is $p = -1+\sqrt{2}$

Desempate 2 Una parábola cuyas raíces se encuentran en A (–3, 0) y B (6, 0) tiene el intercepto en y en el punto $C(0, p)$, $p < 0$, tal que el ángulo ACB es un ángulo recto. Si la ecuación de la parábola es: $y = ax^2 + bx + c$, determine el valor de $(abc)^2$.

A parabola whose roots are at A (–3, 0) and B (6, 0) has the y-intercept at point C(0, p), p < 0, such that the angle ACB is a right angle. If the equation of the parabola is $y = ax^2 + bx + c$, *compute the value of* $(abc)^2$.

Solution:

Since thre roots are -3 and 6

$y = a(x+3)(x-6) = ax^2 - 3ax - 18a$

Using the Pythagorean theorem with the distance formula:

$$\left(\sqrt{9+p^2}\right)^2 + \left(\sqrt{36+p^2}\right)^2 = 9^2$$

From which we get: $p = 3\sqrt{2}$

Now

$p = c = -18a$

$3\sqrt{2} = -18a$

$$a = \frac{-\sqrt{2}}{6}$$

$$b = -3a = \frac{\sqrt{2}}{2}$$

Hence

$$\left(abc\right)^2 = \frac{1}{2}$$