4. DYNAMIC MODELING THEORY

4.1. INTRODUCTION

One of the main objectives of the development of the earthquake simulator facility is testing and analysis of small-scale dynamic models. The purpose of model analysis in earthquake engineering is the prediction of the dynamic response of prototype structures from laboratory tests on physical models [4]. Prior to the discussion of the design of the shaking table components it is necessary a brief discussion of *Dynamic Modeling Theory*, given that many design concepts depend on or are related to its theory. This brief discussion of *Dynamic Modeling Theory* is based on the work titled "Theory and Application of Experimental Model Analysis in Earthquake Engineering" by Moncarz [4] and on of the work titled "Analytical Modeling and Experimental Identification of a Uniaxial Seismic Simulator" by Twitchell [19].

4.2. MODELING THEORY

Modeling Theory establishes how the properties of the model and the properties of the prototype are related. Some of these properties include geometry, material properties, initial conditions, boundary conditions and loading. To obtain a set of correlation or scaling laws for the model-prototype correspondence it is necessary to use *Similitude Theory* which can be developed by *Dimensional Analysis*.

4.3. DIMENSIONAL ANALYSIS

Almost all physical phenomena can be described through mathematical expressions or equations. Dimensional analysis is developed from considering these expressions and paying attention to the significant quantities involved in them and the dimensions that describe these quantities. This analytical tool starts from the premise that every physical phenomenon can be expressed by a dimensionally homogenous equation of the type [4]:

$$q_1 = F\left(q_2, q_3, \ldots q_n\right) \tag{4.1}$$

where n is the total number of physical quantities involved in the expression describing the phenomena, q_1 is a dependent quantity and q_2 to q_n are the variables and parameters on which q_1 depends [4]. According to Buckingham's Pi Theorem [4]:

> "a dimensionally homogenous equation can be reduced to a functional relationship between a complete set of independent dimensionless products (π-factors)."

Therefore Equation 4.1 can be written in the form [4]:

$$p_1 = f\left(p_2, p_3, \ldots p_{n-N}\right) \tag{4.2}$$

where π_1 to π_{n-N} are dimensionless products of powers of the physical quantities q_1 to q_n. The number N is the rank of the dimensional matrix which is usually equal to the number of basic units needed to describe the physical quantities [4]. In engineering, the most common set of basic quantities are those of mass (M), length (L), time (T), and temperature (θ) or force (F), L, T and θ.

Since Equations 4.1 and 4.2 are the same, they describe the same physical phenomenon and, because the dimensionless form of Equation 4.2, it must be equal in the prototype and model if complete similitude is to be attained. Therefore, for complete similitude [4]:

$$\left(p_1\right)_p = \left(p_1\right)_m \tag{4.3}$$

and

$$\left(p_2\right)_p = \left(p_2\right)_m$$

$$\vdots$$

$$\left(p_{n-N}\right)_p = \left(p_{n-N}\right)_m \tag{4.4}$$

Equation 4.3 is the prediction equation and Equations 4.4 constitute the design conditions for the model. Methods for deriving the dimensionless products are discussed at depth by Moncarz [4]. It is important to know, though, that the number of *N* independent dimensionless products is equal to the total number *n* of physical quantities involved minus the number *N* of fundamental quantities needed to describe the dimensions of all physical quantities [4]. Some of the dimensionless products that are most frequently used in engineering and are commonly used in defining physical problems are shown in Table 4.1. The physical variables in the table are: ρ = mass density, v = velocity, L = length, ν = Poisson's ratio, E = modulus of elasticity, σ = stress, P = pressure, d = displacement, t = time and g = acceleration of gravity.

4.4. SIMILITUDE RELATIONSHIPS AND TYPES OF MODELS

Following Moncarz [4], the procedure to find the necessary conditions for complete similitude between model and prototype can be summarized in the following procedure:

1. Write down all physical quantities on which the solution of the physical phenomena under study depends significantly.
2. Develop a complete set of independent dimensionless products from these physical quantities (Eq. 4.2).
3. Establish equality between prototype and model for each of the independent dimensionless products (Eq's. 4.2 and 4.4).

This last step establishes the scaling laws for all physical quantities or products of physical quantities for the physical phenomena. These scaling laws are expressed as ratios of the numbers of units needed to describe identical quantities in model versus prototype. For example, the length scale factor is defined as follows:

$$l_L = \frac{L_p}{L_m} = \frac{\text{Length of Prototype}}{\text{Length of Model}} \tag{4.5}$$

Table 4.1 Dimensionless Products.[1]

Named Dimensionless Product	Formula
Cauchy Number	$\frac{r v^2}{E}$
Froude Number	$\frac{v^2}{Lg}$
Reynolds Number	$\frac{Lv}{u}$
Dimensionless Products Commonly Encountered in Structural Engineering Problems	
$\frac{r v^2}{E}, \frac{v^2}{Lg}, \frac{sL^2}{P}, \frac{EL^2}{P}, t\sqrt{\frac{a}{L}}, aT$	
$\frac{r a L^3}{P}, \frac{t}{L}\sqrt{\frac{E}{P}}, \frac{r g L}{E}, \frac{s}{E}, \frac{d}{L}, \frac{a}{g}$	

Note: 1. Modified from [4].

A model that fulfills all similitude requirements is called a "true replica model". In many practical situations the fulfillment of all design conditions will be an impossible task. These kinds of models can be classified as "adequate" or "distorted models".

"Adequate models" are those where the prediction equation is not affected and the design condition may be violated when insight into physical problem reveals that the results will not depend significantly on the violated design condition [4].

Distorted models are those where the distortion in one dimensionless product either leads to a distortion of the prediction equation or is accounted for by introducing compensating distortions in other dimensionless products [4].

4.5. PHYSICAL MODELS FOR SHAKE TABLE STUDIES

4.5.1. TRUE REPLICA MODELS

As stated earlier, true replica models must satisfy all similitude requirements. Let us assume that we want to reproduce at model scale the time history of stress components $s_{ij}\left(\vec{r},t\right)$ in a replica model subjected to an acceleration time history vector *a* (t). Since the distributions of stress and of material in the prototype and model must be the same, *Dimensional Analysis* can be applied [4]. Let's call σ a typical stress, ρ a typical density, and E a representative stiffness property of the material. The typical stress can be expressed through a functional relationship of the form [4]:

$$s = F\left(\vec{r}, t, r, E, a, g, L, s_o, \vec{r}_o\right) \tag{4.6}$$

where σ_o and $\vec{r}_o$ refer to initial conditions. In this expression it is assumed a similarity of material between prototype and model.

Following *Dimensional Analysis*, a complete set of dimensionless products is generated from the dimensional matrix of the quantities in Equation 4.6 [4].

$$\frac{s}{E} = f\left(\frac{\vec{r}}{L}, \frac{t}{L}\sqrt{\frac{E}{r}}, \frac{a}{g}, \frac{gLr}{E}, \frac{s_o}{E}, \frac{\vec{r}_o}{L}\right) \tag{4.7}$$

Since the gravitational acceleration can not be changed between model and prototype, the value of λ_g must be taken equal to one. Therefore, from the dimensionless product a/g (Froude's Number, usually written as v^2/Lg) it follows that [4]:

$$\lambda_a = \lambda_g = 1 \tag{4.8}$$

The ratio of the modulus of elasticity, E, to the specific weight, γ, is called the specific stiffness of the material. This ratio is taken from the dimensionless product $(gL\rho/E)_r$, where $\rho g = \gamma$ [4]. For a true replica model, the specific stiffness scale factor, $\lambda_{E/\gamma}$, must be satisfied. Using *Dimensional Analysis*, the specific stiffness scale factor may be determined as follows [19]:

$$l_{\frac{E}{g}} = \frac{\left(\frac{E}{g}\right)_p}{\left(\frac{E}{g}\right)_m} = \frac{\frac{\left(\frac{F}{L^2}\right)_p}{\left(\frac{F}{L^3}\right)_p}}{\frac{\left(\frac{F}{L^2}\right)_m}{\left(\frac{F}{L^3}\right)_m}} = \frac{L_p}{L_m} = l_L \tag{4.9}$$

where F is force, L is length, and p and m distinguishes parameters of the prototype and model, respectively. From Equation 4.9, it can be seen that since λ_L must be greater than unity the specific stiffness of the model must be less than the specific stiffness of the prototype. This scaling law places a severe limitation on the choice of suitable model materials.

It is often desirable to construct the model of the same material as the prototype. In this case, the modulus of elasticity scale factor, λ_E, will be equal to unity and Equation 4.9 reduces to [19]:

$$l_{\frac{E}{g}} = \frac{\left(\frac{E}{g}\right)_p}{\left(\frac{E}{g}\right)_m} = \frac{l_E}{l_g} = \frac{1}{l_g} = l_g^{-1} = l_L \tag{4.10}$$

From Equation 4.10, the specific weight at model scale can be written as [19]:

$$g_m = l_L g_p \tag{4.11}$$

this shows that the model's material must have a larger specific weight than the prototype to comply with the true replica model similitude requirements [19]. True replica models are extremely difficult to realize because of problems in material simulation. But it is possible to deal with this problem through artificial mass simulation.

4.5.2. Adequate Models

Adequate models are physical models that although violate one dimensionless product the distortion does not affect other dimensionless products or the prediction equation. The need for such models is based on the desire to use the same materials as in prototypes [4].

I. Model Tests with "Artificial" Mass Simulation [19]

As it has been shown above, if both the prototype and model are constructed of the same material, the specific weight of the model material must be larger than the specific weight of the prototype material [19]. But, since the same material is being used for both the prototype and the model ($\lambda_\rho = 1$), and the prototype and the model are subjected to the same gravitational acceleration (λ_g = 1), the specific weight scale factor will be unity (i.e., λ_γ =1). The solution to this problem lies in augmenting, the specific weight of the structurally effective material with additional material which is structurally not effective [4]. An example on how to determine the required amount of additional mass necessary to meet the specific weight similitude requirement is described below [19].

Let's consider a reduced-scale model which is constructed of the same material as the prototype and is subjected to the same gravitational accelerations (i.e., $\lambda_g = \lambda_\rho = \lambda_\gamma = \lambda_E$ = 1) [19]. The mass scale factor provided in this case is:

$$\lambda_M^{prov} = \frac{M_p}{M_m^{prov}} = \lambda_\rho \lambda_L^3 = 1 \cdot \lambda_L^3 = \lambda_L^3 \tag{4.12}$$

The required mass scale factor for true replica model is:

$$\lambda_M^{reqd} = \frac{M_p}{M_m^{reqd}} = \frac{\lambda_\gamma}{\lambda_g} \lambda_L^3 = \frac{\lambda_L^{-1}}{1} \lambda_L^3 = \lambda_L^2 \tag{4.13}$$

Equation 4.10 was used to reduce λ_γ with λ_L^{-1}. It can be seen from Equation 4.12 and 4.13 that the provided mass of the model, M_m^{prov}, is less than the required mass of the model, M_m^{reqd}. Therefore, additional mass must be added to the model structure to meet the specific stiffness requirement. The required additional mass ΔM is determined as follows [19]:

$$\lambda_M^{prov} = \frac{M_p}{M_m^{prov}} \tag{4.14}$$

$$M_m^{prov} = M_p \left(\lambda_M^{prov} \right)^{-1} = M_p \lambda_L^{-3} \tag{4.15}$$

$$l_M^{reqd} = \frac{M_p}{M_m^{reqd}} \tag{4.16}$$

$$M_m^{reqd} = M_p \left(l_M^{reqd} \right)^{-1} = M_p \, l_L^{-2} \tag{4.17}$$

$$\Delta M = M_m^{reqd} - M_m^{prov} = M_p \left(l_L^{-2} - l_L^{-3} \right) \tag{4.18}$$

Equation 4.18 gives the required additional mass in terms of the mass of the prototype structure [19].

The artificial mass simulation method involves the addition of structurally not effective mass to augment the specific weight of the model structure. The method is particularly well-suited to lumped-mass models such as shear-type buildings, where the mass may be easily concentrated at discrete locations (e.g., at the floor levels) [4, 19].

Utilizing the method described above, the design of the model structure begins with the selection of values for *N* scale factors [19]. This scale factors are taken from Table 4.2. For seismic testing, the basic dimensions may be taken as force, length, and time, and thus N = 3. In the artificial mass simulation method in which the same materials are used in the model and prototype, $\lambda_g = \lambda_E = 1$. The designer must select the last scale factor which is usually the value of λ_L. All other quantities can be expressed in terms of these three scale factors, as shown in Table 4.2 [19].

Table 4.2 Similitude Relationships for Artificial Mass Simulation Method.[1]

Parameter	Units[2]	Any Material	Same Material as Prototype
Length	L	l_L	l_L
Time	T	$l_L^{1/2}$	$l_L^{1/2}$
Frequency	$\frac{1}{T}$	$l_L^{-1/2}$	$l_L^{-1/2}$
Velocity	$\frac{L}{T}$	$l_L^{1/2}$	$l_L^{1/2}$
Displacement	L	l_L	l_L
Gravitational Acceleration	$\frac{L}{T^2}$	1	1
Acceleration	$\frac{L}{T^2}$	1	1
Force	F	$l_E l_L^2$	l_L^2
Mass	$\frac{F \cdot T^2}{L}$	$l_E l_L^2$	l_L^2
Specific Stiffness	L	l_L	l_L
Strain	$\frac{L}{L}$	1	1
Stress	$\frac{F}{L^2}$	l_E	1
Modulus of Elasticity	$\frac{F}{L^2}$	l_E	1
Energy	FL	$l_E l_L^3$	l_L^3

Notes:

1. From [19].
2. L = Length, T =Time, F = Force and E = Modulus of Elasticity.